PHYS 320 ANALYTICAL MECHANICS

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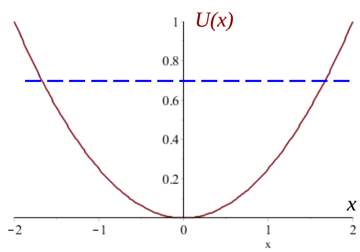
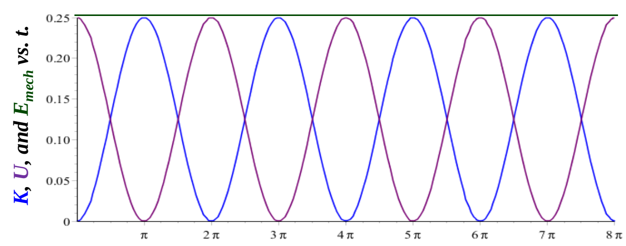
Simple Harmonic Oscillations (no damping)

Energy considerations $x(t) = A\cos(\omega_o t - \phi)$



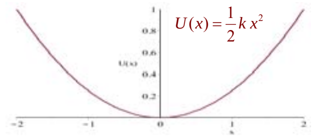
$$E_{mech}(t) = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2 = \frac{1}{2} kA^2$$



$$\omega_o \equiv \sqrt{\frac{k}{m}}$$

Simple Harmonic Oscillations (no damping)

Energy considerations for a particle oscillating about a point of stable equilibrium.



Any potential well can be modeled as approximately parabolic for small enough oscillations

• Can do a Taylor series expansion about equilibrium position, x_o:

$$U(x)=\underbrace{U(x_o)}_{\text{const!}}+\underbrace{\left.\frac{dU(x)}{dx}\right|_{x_o}}_{=0}(x-x_o)+\underbrace{\frac{1}{2!}\left.\frac{d^2U(x)}{dx^2}\right|_{x_o}(x-x_o)^2}_{\text{looks like }\frac{1}{2}kx^2}+\underbrace{\ldots}_{\text{ignore small higher order terms!}}$$

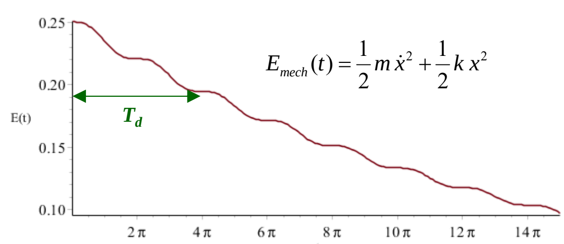
• with $u = x - x_o$ and with $u < 1$, we can write

$$\boxed{\omega=\sqrt{\frac{k}{m}}=\sqrt{\frac{1}{m}\left.\frac{d^2U(x)}{dx^2}\right|_{\substack{u=0\\x=x_o}}}}$$

for small oscillations about equilibrium

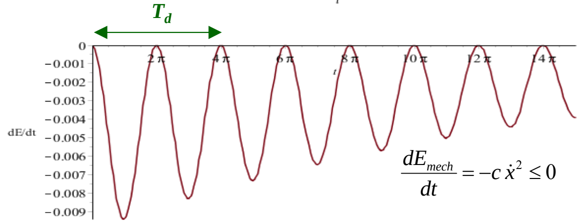
Damped Oscillations (linear damping)

Energy considerations $x(t)=Ae^{-\gamma t}\cos(\omega_d t-\phi)$

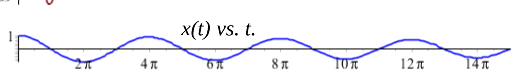


$$E_{mech}(t)=\frac{1}{2}m\dot{x}^2+\frac{1}{2}kx^2$$

Mechanical energy dissipated as frictional heat



$$\frac{dE_{mech}}{dt}=-c\,\dot{x}^2\le 0$$



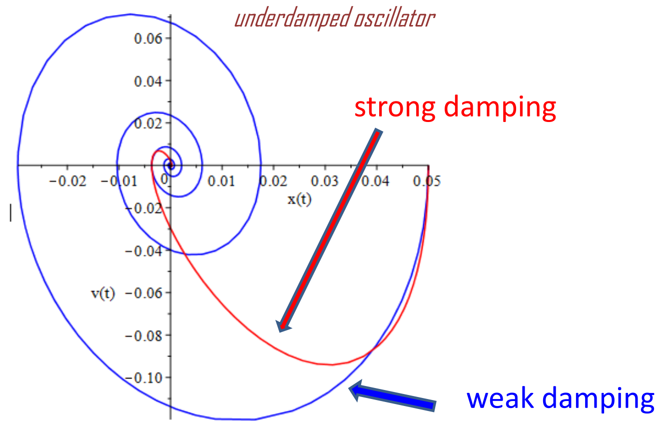
$$\boxed{\omega_d^2\equiv\omega_o^2-\gamma^2}$$

$$\boxed{\omega_o\equiv\sqrt{\frac{k}{m}}}$$

$$\boxed{\gamma\equiv\frac{c}{2m}}$$

Phase plots

- For a 1-D oscillator, the motion is completely specified by two quantities: $x(t)$ and $\dot{x}(t)$



- Note that

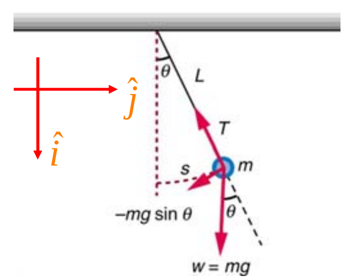
$$E_{mech} = K + U = \frac{1}{2} m\,\dot{x}^2 + \frac{1}{2} k\,x^2$$

$$= \frac{1}{2} m\,(-A\omega_o e^{-\gamma t}\sin(\omega_o t - \phi) - A\gamma e^{-\gamma t}\cos(\omega_o t - \phi))^2 + \frac{1}{2} k\,(Ae^{-\gamma t}\cos(\omega_o t - \phi))^2$$

$$= \frac{A^2}{2} e^{-2\gamma t}\left[m\,\omega_o^2 \sin^2(\omega_o t - \phi) + 2m\omega_o\gamma\sin(\omega_o t - \phi)\cos(\omega_o t - \phi) + m\gamma^2\cos^2(\omega_o t - \phi) + k\cos^2(\omega_o t - \phi)\right]$$

$$= \frac{A^2}{2} e^{-2\gamma t}\left[1 + 2m\omega_o\gamma\sin(\omega_o t - \phi)\cos(\omega_o t - \phi) + m\gamma^2\cos^2(\omega_o t - \phi)\right]$$

The Simple Pendulum



- Relative to the lowest point in the swing of the bob: $U(\vec{r}) = mgL(1 - \cos\theta)$

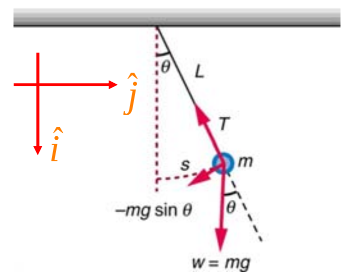
$$\therefore\ \vec{F} = -\vec{\nabla} U = -\hat{r}\frac{\partial U}{\partial r} - \hat{\theta}\frac{1}{L}\frac{\partial U}{\partial \theta} - \hat{k}\frac{\partial U}{\partial z} = -\hat{\theta}\, m\, g \sin\theta$$

Note that del operator takes on different forms in different coordinate systems (see inside back cover of textbook!); here, r = L

$$\therefore\ \vec{F} = m\ddot{\vec{r}} = m(\hat{r}(\cancelto{0}{\ddot{r}} - r\dot{\theta}^2) + \hat{\theta}(r\ddot{\theta} + \cancelto{0}{2\dot{r}}\dot{\theta}) + \hat{k}(\cancelto{0}{\ddot{z}})) = \hat{r}\, m\ f(\theta) - \hat{\theta}\, m\, g\sin\theta$$

this *centripetal* term has no potential associated with it

The Simple Pendulum



- The angular component of the equation of motion is then

$$mL\ddot{\theta} = -m\,g\sin\theta$$

$$\therefore\ \ddot{\theta} = -(g/L)\sin\theta$$

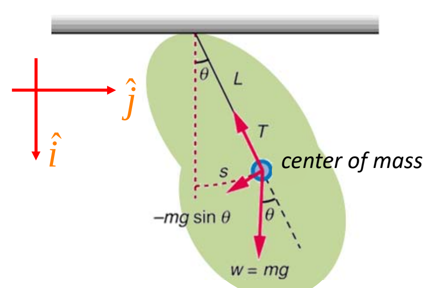
- For small angles, $\ddot{\theta} = -(g/L)\theta$

  We identify $\omega_o^2 = (g/L)$

Can also use:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{1}{m}\frac{d^2U(x)}{dx^2}\bigg|_{\substack{u=0\\x=x_o}}}$$

The Physical Pendulum



- Another approach:

$$\vec{\tau}_{net} = I\vec{\alpha} \Rightarrow I\ddot{\theta} = -m\,g\,L\sin\theta$$

$$\therefore\ \ddot{\theta} = -(mgL/I)\sin\theta$$

- For small angles

$$\ddot{\theta} = -(mgL/I)\theta$$

We identify $\omega_o^2 = (mgL/I)$